Section 2.3
The Basic Limit Laws

(1) The Limit Laws

(2) Examples

(3) Assumptions Matter

Basic Limit Laws:

Assume that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ each **exist**.

Identity and Constant Laws $\lim_{x\to c} x = c$ $\lim_{x\to c} 1 = 1$

Sum Law $\lim_{x\to c} (f(x)+g(x)) = \left(\lim_{x\to c} f(x)\right) + \left(\lim_{x\to c} g(x)\right)$

Constant Multiple Law $\lim_{x\to c} (kf(x)) = k\left(\lim_{x\to c} f(x)\right)$

Product Law $\lim_{x\to c} (f(x)g(x)) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$

Quotient Law

If $\lim_{x\to c} g(x) \neq 0$,

$$\lim_{x\to c} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$$

Power Law

If n is an integer,

$$\lim_{x\to c} (f(x)^n) = \left(\lim_{x\to c} f(x)\right)^n$$

(I) $\lim_{x\to c} x^3+4x^2-3$ (II) $\lim_{x\to c} \frac{x^4+x^2-1}{x^2+5}$ (III) $\lim_{x\to c} \frac{x^2-2x-15}{x^2-9}$

Example IV

$$\lim_{x\to -1} f(x) = 3 \qquad \lim_{x\to -1} g(x) = -2$$

$$\lim_{x\to 2} f(x) = -1 \qquad \lim_{x\to 2} g(x) = 4$$

With the above limit information, evaluate the limits:

(i) $\displaystyle\lim_{x\to -1} (2f(x) - 3g(x)) = 2\cdot 3 - 3\cdot(-2) = 12$

(ii) $\displaystyle\lim_{x\to 2} \frac{x\sqrt{g(x)}}{f(x)^2} = \frac{2\sqrt{4}}{(-1)^2} = 4$

(iii) $\displaystyle\lim_{x\to -1} \frac{g(-2x)}{x^2}$???? *(Solved in section 2.4!)*

Assumptions Matter

Every Basic Limit Law rests upon the assumption that the **limits exist**! If either limit fails to exist, the limit laws cannot be applied:

(Example V) The Product Law cannot be applied to $\lim_{x \to 0} f(x)g(x)$ if $f(x) = x$ and $g(x) = x^{-1}$.

For all $x \neq 0$, we have $f(x)g(x) = x \cdot x^{-1} = 1$, so the limit of the product exists:

$$\lim_{x \to 0} f(x)g(x) = \lim_{x \to 0} 1 = 1$$

However, $\lim_{x \to 0} x^{-1}$ does **not** exist! Therefore, it is **incorrect** to use the Product Law to evaluate the limit. In this case,

$$\lim_{x \to 0} f(x)g(x) \neq \left(\lim_{x \to 0} f(x)\right)\left(\lim_{x \to 0} g(x)\right)$$

Assumptions Matter

Every Basic Limit Law rests upon the assumption that the limits **exist**! If either limit fails to exist, the limit laws cannot be applied. However, the combination of limits may exist anyway!

(Example VI) Give an example where $\lim_{x \to c} \dfrac{f(x)}{g(x)}$ exists, but neither $\lim_{x \to c} f(x)$ nor $\lim_{x \to c} g(x)$ exist.

An infinite number of examples exist. The simplest example may be $c = 0$ and $f(x) = g(x) = x^{-1}$.

- $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = \lim_{x \to 0} x^{-1}$ does not exist.
- On the other hand, $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \lim_{x \to 0} 1 = 1$.